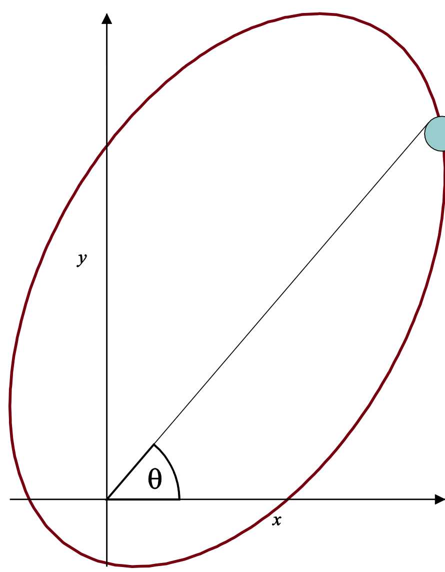
LONG QUESTIONS

1. A moon is orbiting a planet such that the orbit is perpendicular to the surface of the planet where an observer is standing. After some necessary scaling, suppose the orbit satisfies the following equation

$$9\left(\frac{x}{2}+\frac{\sqrt{3}y}{2}-4\right)^2+25\left(-\frac{\sqrt{3}x}{2}+\frac{y}{2}\right)^2=225$$

Let r be the radius of the moon. Assume that the period of rotation of the planet is much larger than the orbital period of the moon. Determine $\tan\frac{\theta}{2}$, where θ is the elevation angle when the moon looks largest to the observer.



2. Two massive stars A and B with masses m_A and m_B, respectively, are separated by a distance *d*. Both stars orbit around their center of mass under gravitational force. Suppose their orbits are circular and lie on the *X*-*Y* plane whose origin is at the stars' center of mass (see Figure 2)
   a. Calculate the tangential and angular speeds of star A.

   An observer standing on the *Y*-*Z* plane (see Figure 2) sees the stars from ~~the~~ a large distance with angle θ relatively to the *Z*-axis. He measures that the velocity component of start A along the line of his sight has the form of $K\,cos(\omega t+\varepsilon)$, with *K* and ε being positive constants.

   b. Express the value of $K^3/\omega G$ in terms of m_A, m_B , and θ , where *G* is the universal gravitational constant.

Suppose that the observer can then identify that the star A has the mass equal to $30M_S$ where M_S is the Sun's mass. In addition, he observes that the star B produces X-ray~~s~~, from which he can use the information to classify whether the star B is a neutron star or a black hole. The conclusion depends on the value of m_B, that is: 1) If $m_B < 2M_S$, then B is a neutron star. 2) If $m_B > 2M_S$, then B is a black hole.

c. A measurement has been done by the observer that gives $\frac{K^3}{\omega G} = \frac{1}{250} M_S$. In practice, the value of θ is usually not known. Assuming that the value of θ is equally probable for all possible θ values, calculate the probability of star B to be a black hole. (Hint: Use $\int \sin x \; dx = -\cos x + C$)

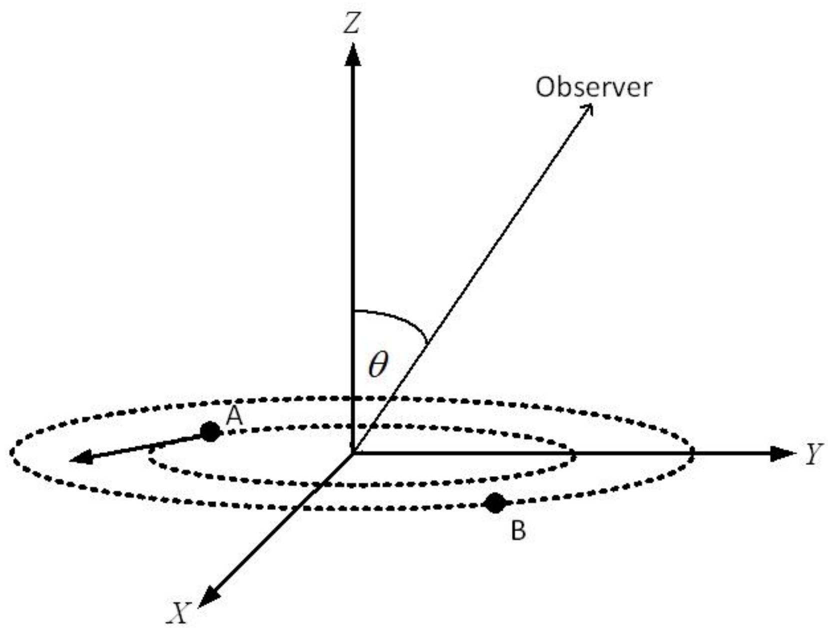


Figure 2

3. Suppose a static spherical star consists of *N* neutral particles with radius *R* (see Figure 1).

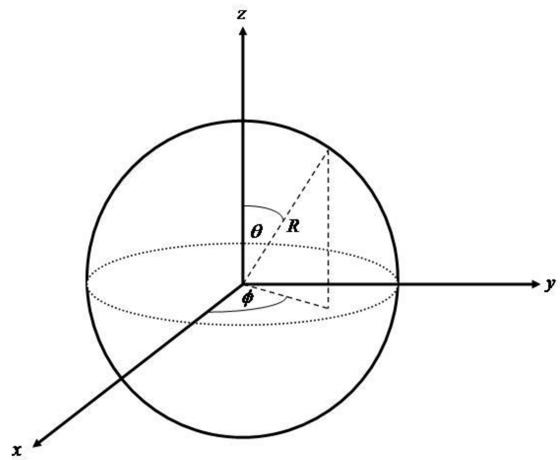


Figure 1

with $\leq \boldsymbol{\theta} \leq \boldsymbol{\pi}$, $\mathbf{0} \leq \boldsymbol{\phi} \leq \mathbf{2}\boldsymbol{\pi}$, satisfying the following equation of states

$$\boldsymbol{P\,V = N\,k}\,\frac{\boldsymbol{T_R - T_0}}{\ln(\boldsymbol{T_R/T_0})} \tag{1}$$

where *P* and *V* are the pressure inside the star and the volume of the star, respectively, and *k* is Boltzmann constant. $\boldsymbol{T_R}$ and $\boldsymbol{T_0}$ are the temperature at the surface $\boldsymbol{r = R}$ and the temperature at the center $\boldsymbol{r = 0}$, respectively. Assume that $\boldsymbol{T_R \leq T_0}$.

a. Simplify the stellar equation of states (1) if $\Delta\boldsymbol{T = T_R - T_0 \approx 0}$ (this is called ideal star) (Hint: Use the approximation $\mathbf{ln}(\boldsymbol{1 + x}) \approx \boldsymbol{x}$ for small *x*)

Suppose the star undergoes a quasi-static process, in which it may slightly contracts or expands, such that the above stellar equation of states (1) still holds.

b. Find the work of the star when it expands from V_1 to V_2 in isothermal process where $\boldsymbol{T_R}$ and $\boldsymbol{T_0}$ are constants.

The star satisfies first law of thermodynamics

$$\boldsymbol{Q = \Delta M c^2 + W} \tag{2}$$

where Q, M, and W are heat, mass of the star, and work, respectively, while c is the light speed in the vacuum and $\boldsymbol{\Delta M} \equiv \boldsymbol{M}_{\mathbf{final}} - \boldsymbol{M}_{\mathbf{initial}}$.

In the following we assume $\boldsymbol{T_0}$ to be constant, while $\boldsymbol{T_R} \equiv \boldsymbol{T}$ varies.

c. Find the heat capacity of the star at constant volume $\boldsymbol{C_v}$ in term of M and at constant pressure C_p expressed in C_v and T (Hint: Use the approximation $(1+x)^n \approx 1+nx$ for small x)

Assuming that $\boldsymbol{C_v}$ is constant and the gas undergoes the isobar process so the star produces the heat and radiates it outside to the space.

d. Find the heat produced by the isobar process if the initial temperature and the final temperature are $\boldsymbol{T_i}$ dan $\boldsymbol{T_f}$, respectively.

e. Suppose there is an observer far away from the star. Related to point d., estimate the distance of the observer if the observer has 0.1% error in measuring the effective temperature around the star.

Now we take an example that the star to be the Sun of the mass $\boldsymbol{M_\odot}$, its radius $\boldsymbol{R_\odot}$, its luminosity (radiation energy emitted per unit time) $\boldsymbol{L_\odot}$, and the Earth-Sun distance, $\boldsymbol{d_\odot}$.

f. If the sunlight were monochromatic with frequency $\mathbf{5} \times \mathbf{10^{14}}$ Hz, estimate the number of photons radiated by the Sun per second.

g. Calculate the heat capacity $\boldsymbol{C_v}$ of the Sun assuming its surface temperature runs from 5500 K until 6000 K in this period.