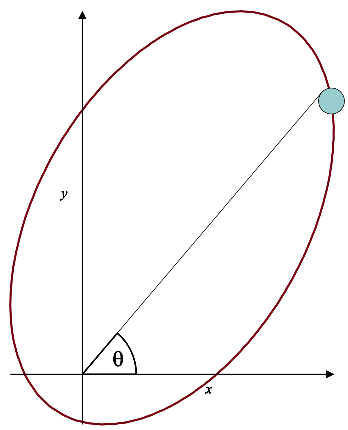
LONG QUESTIONS: MARKING SCHEME

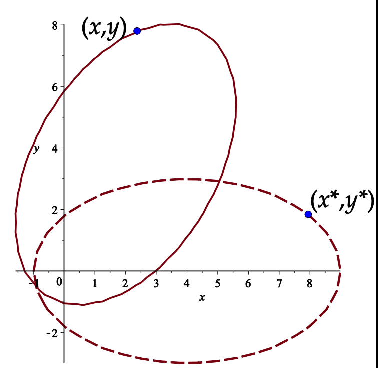
1. A moon is orbiting a planet such that the orbit is perpendicular to the surface of the planet where an observer is standing. After some necessary scaling, suppose the orbit satisfies the following equation

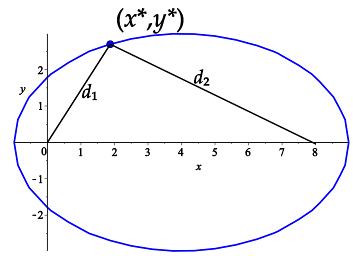
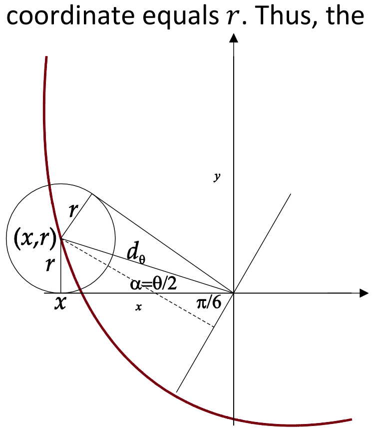
$$9\left(\frac{x}{2}+\frac{\sqrt{3}y}{2}-4\right)^2+25\left(-\frac{\sqrt{3}x}{2}+\frac{y}{2}\right)^2=225$$

Let r be the radius of the moon. Assume that the period of rotation of the planet is much larger than the orbital period of the moon. Determine $\tan\frac{\theta}{2}$, where θ is the elevation angle when the moon looks largest to the observer.



Answer and Marking Scheme:

1	**Notice the standard version of the orbits** The ellipse may be obtained from a standard ellipse $$\frac{(x^*-4)^2}{25}+\frac{(y^*)^2}{9}=1$$ by rotating the standard ellipse with respect to the origin, counterclockwise, (by $\pi/3$ radians). Then $c=\sqrt{a^2-b^2}=\sqrt{16}=4$	 	**10**

2	**Established relation between coordinates before and after transformation.** For any point (x, y) on the ellipse, let (x^*, y^*) be its coordinates before the transformation. From the equation of the ellipse, we can easily see that  $$x^* = \frac{x}{2} + \frac{\sqrt{3}y}{2}$$ $$y^* = -\frac{\sqrt{3}x}{2} + \frac{y}{2}$$ (In fact, $\begin{bmatrix} x^* \\ y^* \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{x}{2} + \frac{\sqrt{3}y}{2} \\ -\frac{\sqrt{3}x}{2} + \frac{y}{2} \end{bmatrix}$.)	20
3	**Obtain expression for calculating distance from a point on an ellipse to the foci.** Consider standard ellipse $$\frac{(x^* - 4)^2}{5^2} + \frac{(y^*)^2}{3^2} = 1$$ Choose any point (x^*, y^*) on the ellipse. Let d_1 and d_2 be the distances from any point (x^*, y^*) on the ellipse to the foci $(0,0)$ and $d(8,0)$; $d_1^2 = (x^*)^2 + (y^*)^2$ and $d_2^2 = (x^* - 8)^2 + (y^*)^2$ Thus, $$d_1^2 = (x^*)^2 + (y^*)^2 = (x^*)^2 + \left(9 - \frac{9(x^* - 4)^2}{5^2}\right) = \frac{(16(x^*)^2 + 72x^* + 81)}{25}$$	30
4	**Identify and characterized the coordinates when the planet looks largest.** The following is the position of the moon when it looks largest. At that point, its second coordinate equals r. Thus, the coordinates be (x, r).  	20
5	**Solve equation $\frac{(16(x^*)^2 + 72x^* + 81)}{25} = x^2 + r^2$ to obtain value of x.** Therefore $$d_1^2 = \frac{(16(x^*)^2 + 72x^* + 81)}{25} = x^2 + r^2$$	15

	where $x^* = \frac{x}{2} + \frac{r\sqrt{3}}{2}$. Therefore, we can obtain the value of x by solving the above equation for x. Substituting $x^* = \frac{x}{2} + \frac{r\sqrt{3}}{2}$ to the equation, we have $$25d_1^2 = 25x^2 + 25r^2 = 16\left(\frac{x}{2} + \frac{r\sqrt{3}}{2}\right)^2 + 72\left(\frac{x}{2} + \frac{r\sqrt{3}}{2}\right) + 81 =$$ or $$25x^2 + 25r^2 = 4x^2 + \left(8r\sqrt{3} + 36\right)x + \left(12r^2 + 36\sqrt{3}r + 81\right)$$ $$21x^2 - \left(8r\sqrt{3} + 36\right)x + \left(13r^2 - 36r\sqrt{3} - 81\right) = 0$$ Then use quadratic formula to obtain x in term of r. Choose smaller value of x to get larger value of $\tan\frac{\theta}{2} = \frac{r}{x}$. The smaller one is $$x = \frac{4r\sqrt{3} + 18}{21} - \frac{15\sqrt{-r^2 + 4r\sqrt{3} + 9}}{21}$$	
6	**Find the expression of** $\tan\frac{\theta}{2}$. Hence $$\tan\frac{\theta}{2} = \frac{r}{x} = \frac{21r}{4r\sqrt{3} + 18 - 15\sqrt{-r^2 + 4r\sqrt{3} + 9}}$$	**5**

2. Two massive stars A and B with mass m_A and m_B are separated by a distance *d*. Both stars are orbiting each other with respect to their center of gravity whose orbits are circular. Suppose the stars lie on the *X-Y* plane (see Figure 2) and are moving under gravitational force.

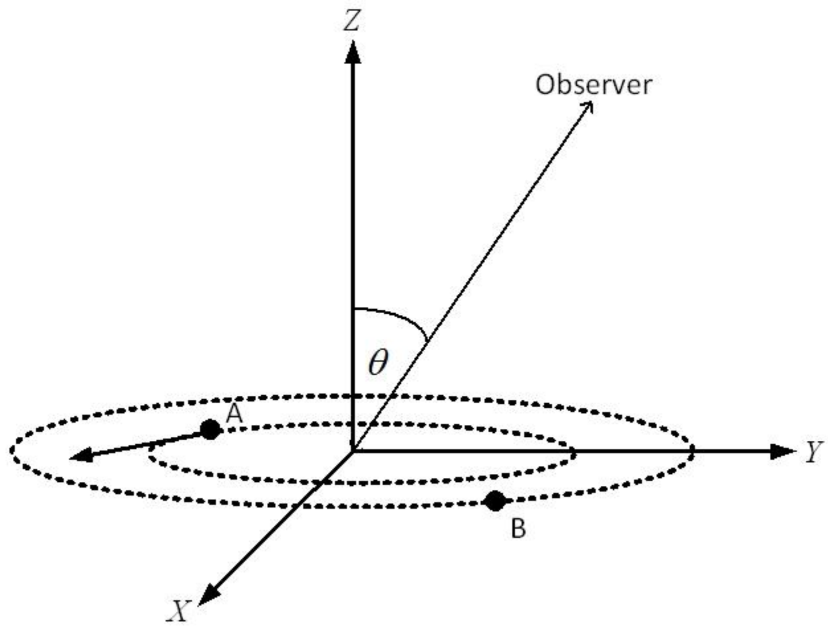


Figure 2

a. Calculate the speed of star A and its angular velocity.

An observer lies on the *Y-Z* plane (see Figure 2) see the stars from the large distance with angle θ relatively to the *Z*-axis. He/she measure that the velocity component of A to the line of his/her sight has the form $K\ cos(\omega t + \varepsilon)$, with *K* and ε are positive.

b. Express the value $K^3/\omega G$ in term of m_A, m_B , and θ where *G* is the universal gravitational constant.

The observer can then identify that the star A has mass equal to $30M_S$ where M_S is the Sun's mass. On the other hand, he/she observe that the star B produces X-rays, so it could be a neutron star or a black hole. These situations depend on m_B: 1) If $m_B < 2M_S$, then B is a neutron star. 2) If $m_B > 2M_S$, then B is a black hole.

c. A measurement has been done by the observer which results $\frac{K^3}{\omega G} = \frac{1}{250} M_S$. If the value of $\cos\theta$ has the same probaiity, then calculate the probability of B to be a black hole. (Hint: Use $\int \sin x \ dx = -\cos x + C$)

Answer and Marking Scheme:

a.	The center of gravity of the stars is relatively to the star A given by $r_A = \frac{m_A}{m_A+m_B} d$ and since the orbit of A is a circle, then $F_{AX} = \frac{G m_A m_B}{d^2} = \frac{m_A v_A^2}{r_A}$ So, we get $v_A = m_B \sqrt{\frac{G}{(m_A+m_B)d}}$ The angular velocity of A is given by $\omega = \frac{v_A}{r_A} = \sqrt{\frac{G(m_A+m_B)}{d^3}}$)	**30**
b.	In Cartesian coordinate system, the velocity of A is $\vec{v}_A(t) = v_A(-\sin(\omega t+\varepsilon)\,\hat{\imath} + \cos(\omega t+\varepsilon)\,\hat{\jmath})$ Unit vector of the observer is $\hat{r}_P = \cos\theta\,\hat{k} + \sin\theta\,\hat{\jmath}$ so the component of $\vec{v}_A$ in the line of the observer sight is given by $\vec{v}_A \cdot \hat{r}_P = v_A \sin\theta \ \cos(\omega t+\varepsilon)$ Since the component of $\vec{v}_A$ in the line of the observer sight is $K\ cos(\omega t+\varepsilon)$, then $K = v_A \sin\theta$ Finally, we have $\frac{K^3}{\omega G} = \frac{m_B^3}{(m_A+m_B)^2}\sin^3\theta$ (**)	**30**
c.	From the result in b., namely eq. (**), we get	**40**

	$$\sin^3\theta = \frac{K^3}{\omega G}\frac{(m_A+m_B)^2}{m_B^3} < \frac{1}{250}\frac{32^2}{8} = \frac{64}{125}$$ Since $\theta \in [0,\pi]$, then $\sin\theta < 0{,}8$. Thus, the probability of B is a black hole is the same as the probability of $\sin\theta < 0{,}8$ for $\theta \in [0,\pi]$. Since $\sin^{-1} 0{,}8 \approx 53^{\mathrm{o}}$ or 127^{o}, then $$P(\mathrm{B: Black\ Hole}) = 1 - \frac{\int_{53^{\mathrm{o}}}^{127^{\mathrm{o}}} \sin\theta\ d\theta}{\int_{0^{\mathrm{o}}}^{180^{\mathrm{o}}} \sin\theta\ d\theta} = 1 - 0{,}6 = 0{,}4$$	

3. Suppose a static spherical star consists of *N* neutral particles with radius *R* (see Figure 1).

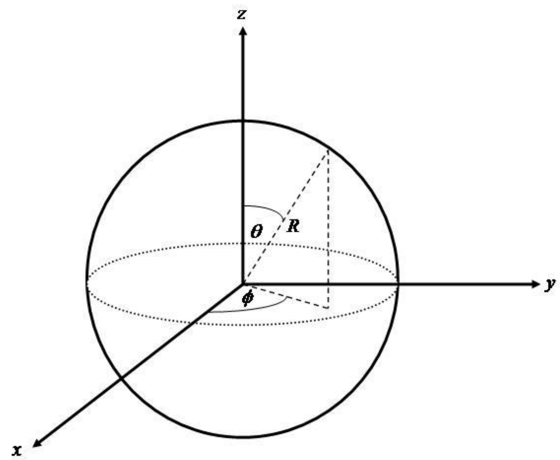


Figure 1

with $\mathbf{0 \le \theta \le \pi}$, $\mathbf{0 \le \phi \le 2\pi}$, satisfying the following equation of states

$$\boldsymbol{P\,V = N\,k\,\frac{T_R - T_0}{\ln(T_R/T_0)}} \tag{1}$$

where *P* and *V* are the pressure inside the star and the volume of the star, respectively, and *k* is Boltzmann constant. $\boldsymbol{T_R}$ and $\boldsymbol{T_0}$ are the temperature at the surface $\boldsymbol{r = R}$ and the temperature at the center $\boldsymbol{r = 0}$, respectively. Assume that $\boldsymbol{T_R \le T_0}$.

a. Simplify the stellar equation of states (1) if $\Delta\boldsymbol{T = T_R - T_0 \approx 0}$ (this is called ideal star) (Hint: Use the approximation $\mathbf{ln}\boldsymbol{(1 + x) \approx x}$ for small *x*)

Suppose the star undergoes a quasi-static process, in which it may slightly contract or expand, such that the above stellar equation of states (1) still holds.

b. Find the work of the star when it expands from $\boldsymbol{V_1}$ to $\boldsymbol{V_2}$ in isothermal process where $\boldsymbol{T_R}$ and $\boldsymbol{T_0}$ are constants.

The star satisfies first law of thermodynamics

$$\boldsymbol{Q = \Delta M c^2 + W} \tag{2}$$

where *Q*, *M*, and *W* are heat, mass of the star, and work respectively, while *c* is the light speed in the vacuum and $\boldsymbol{\Delta M \equiv M_{\mathbf{final}} - M_{\mathbf{initial}}}$.

In the following we assume $\boldsymbol{T_0}$ to be constant, while $\boldsymbol{T_R} \equiv \boldsymbol{T}$ varies.

c. Find the heat capacity of the star at constant volume $\boldsymbol{C_v}$ in term of *M* and at constant pressure $\boldsymbol{C_p}$ expressed in $\boldsymbol{C_v}$ and *T* (Hint: Use the approximation $(\mathbf{1}+\boldsymbol{x})^{\boldsymbol{n}} \approx \mathbf{1}+\boldsymbol{nx}$ for small *x*)

Assuming that $\boldsymbol{C_v}$ is constant and the gas undergoes the isobar process so the star produces the heat and radiates it outside to the space.

d. Find the heat produced by the isobar process if the initial temperature and the final temperature are $\boldsymbol{T_i}$ dan $\boldsymbol{T_f}$, respectively.

e. Suppose there is an observer far away from the star. Related to point d., estimate the distance of the observer if the observer has 0.1% error in measuring the effective temperature around the star.

Now we take an example that the star to be the Sun of the mass $\boldsymbol{M_\odot}$, its radius $\boldsymbol{R_\odot}$, its luminosity (radiation energy emitted per unit time) $\boldsymbol{L_\odot}$, and the Earth-Sun distance, $\boldsymbol{d_\odot}$.

f. If the sunlight were monochromatic with frequency $\mathbf{5} \times \mathbf{10^{14}}$ Hz, estimate the number of photons radiated by the Sun per second.

g. Calculate the heat capacity $\boldsymbol{C_v}$ of the Sun assuming its surface temperature runs from 5500 K until 6000 K in this period.

Answer and Marking Scheme:

a.	Defining $\Delta T = T_R - T_0$ and $\Delta T \approx 0$, we have $$P\,V = N\,k\,\frac{\Delta T}{\ln(1+\Delta T/T_0)} \qquad (3)$$ using $\ln(1+\Delta T/T_0) \approx \Delta T/T_0$, we then obtain $$P\,V = N\,k\,T_0 \qquad (4)$$	**10**
b.	If the gas undergoes isothermal process where T_R and T_0 are constant, then the work has the form $$W = \int_{V_1}^{V_2} P\,dV = N\,k\,\frac{T_R - T_0}{\ln(T_R/T_0)}\ln\left(\frac{V_2}{V_1}\right)$$	**15**

c.	The internal energy of the star is $U = Mc^2$ ($U(T) = M(T)c^2$ for ideal star). Thus, the constant volume heat capacity of the star has the form $$C_v = \left(\frac{\Delta Q}{\Delta T}\right)_V = \left(\frac{\Delta M}{\Delta T}\right)_V c^2$$ for small ΔT. Then, using first law of themodynamics, the constant pressure heat capacity of the star is $$C_p = \left(\frac{\Delta Q}{\Delta T}\right)_P = \left(\frac{\Delta M}{\Delta T}\right)_V c^2 + P\,\frac{\Delta V}{\Delta T} = C_v + P\,\frac{\Delta V}{\Delta T}$$ for small ΔT. Defining $\Delta T = T_2 - T_1$, then $$P\,\Delta V = N\,k\left(\frac{T_1 - T_0 + \Delta T}{\ln\big((T_1 + \Delta T)/T_0\big)} - \frac{T_1 - T_0}{\ln(T_1/T_0)}\right)$$ Using the approximation $$\ln\big((T_1 + \Delta T)/T_0\big) \approx \ln\left(\frac{T_1}{T_0}\right) + \frac{\Delta T}{T_1}$$ $$\left(1 + \frac{\Delta T}{T_1 \ln(T_1/T_0)}\right)^{-1} \approx 1 - \frac{\Delta T}{T_1 \ln(T_1/T_0)}$$ then we have $$P\,\frac{\Delta V}{\Delta T} = \frac{N\,k}{\ln(T/T_0)}\left(1 - \frac{(T - T_0)/T}{\ln(T/T_0)}\right)$$ where $T_1 \equiv T$. Finally, we obtain $$C_p = \left(\frac{\Delta Q}{\Delta T}\right)_P = \left(\frac{\Delta M}{\Delta T}\right)_V c^2 + P\,\frac{\Delta V}{\Delta T} = C_v + \frac{N\,k}{\ln(T/T_0)}\left(1 - \frac{(T - T_0)/T}{\ln(T/T_0)}\right)$$	**25**
d.	Since C_v is constant, the heat produced by the star is given by $$Q_H = C_v\,(T_f - T_i) + P\,\Delta\,V$$ $$= C_v\,(T_i - T_f) + N\,k\left(\frac{T_f - T_0}{\ln(T_f/T_0)} - \frac{T_i - T_0}{\ln(T_i/T_0)}\right)$$	**20**
e.	The error 0,1% , namely $$\frac{T_f - T_{eff}}{T_f} = \frac{1}{1000}$$ then $T_{eff} = \frac{999}{1000}\,T_f$. From Stefan law of black body radiation	**15**

	$$\frac{Q_H}{4\pi r_0^2} = \sigma T_{eff}^4 = \sigma \left(\frac{999}{1000}\right)^4 T_f^4$$ where r_0 is the observer's distance which is given by $$r_0 = \left(\frac{Q_H}{4\pi\sigma}\right)^{1/2} \left(\frac{1000}{999}\right)^2 T_f^{-2}$$	
f.	Energy per second radiated by the Sun $$L_\odot = Nh\nu$$ where *N* is the number of photon. Thus $N = \frac{L_\odot}{h\,\nu} = \frac{3.96\times10^{26}}{6.6261\times10^{-34}\times\mathbf{5}\times\mathbf{10^{14}}} = 1.195 \times 10^{45}$ photons	**7**
g.	Energy per second radiated by the Sun is proportional to mass defect of the Sun $$L_\odot = \Delta M\, c^2$$ Thus, $$C_v = \frac{\Delta M\, c^2}{\Delta T} = \frac{L_\odot}{\Delta T} = \frac{3.96 \times 10^{26}}{6000 - 5500}\ \mathrm{J/K} = 7.92 \times 10^{23}\ \mathrm{J/K}$$	**8**